

# B-mode polarization of the CMB from the second-order photon quadrupole

Nicola Bartolo\*

*Dipartimento di Fisica "G. Galilei", Università di Padova  
and INFN - Sezione di Padova  
via Marzolo 8, I-35131 Padova, Italy*

Sabino Matarrese†

*Dipartimento di Fisica "G. Galilei", Università di Padova  
INFN - Sezione di Padova  
via Marzolo 8, I-35131 Padova, Italy*

Silvia Mollerach‡

*Centro Atómico Bariloche, Av. Bustillo 9500  
8400 Bariloche, Argentina*

Antonio Riotto§

*Département de Physique Théorique, Université de Genève, 24 Quai Ansermet, Genève, Switzerland,  
and INFN - Sezione di Padova via Marzolo 8, I-35131 Padova, Italy*

(Dated: October 22, 2013)

We study a new contribution to the polarization of the Cosmic Microwave Background induced at the epoch of recombination by the second-order quadrupole moment of the photon distribution. At second order in perturbation theory the quadrupole moment is not suppressed by the inverse of the optical depth in the tight coupling limit, as it happens at first order in perturbation theory. We concentrate on the B-mode CMB polarization and find that such a novel contribution constitutes a contamination in the detection of the primordial tensor modes if the tensor to scalar ratio  $r$  is smaller than a few  $\times 10^{-5}$ . The magnitude of the effect is larger than the B-mode due to secondary vector/tensor perturbations and the analogous effect generated during the reionization epoch, while it is smaller than the contamination produced by the conversion of polarization of type E into type B, by weak gravitational lensing. However the lensing signal can be cleaned, making the secondary modes discussed here the actual contamination limiting the detection of small amplitude primordial gravitational waves if  $r$  is below  $\simeq 10^{-5}$ .

PACS numbers: 98.70.Vc, 98.80.Es, 98.80.Cq

DFPD 07/A/05

## I. INTRODUCTION

The hunting for the polarization of the Cosmic Microwave Background (CMB) is one of the major targets of present and future planned experiments since it offers a unique opportunity to gather information about both the early (inflationary) universe and more recent epochs. The polarization is generated by the anisotropic scattering of CMB photons off free electrons. The decomposition into E and B-modes [1, 2] is extremely useful when dealing with the nature of the cosmological perturbations that are responsible for the temperature and polarization anisotropies of the CMB: the B-modes are excited only by vector and tensor perturbations, thus allowing eventually to disentangle their effect from that of the scalar perturbations. One of the fundamental predictions of inflationary models is the generation of a stochastic background of gravitational waves (tensor perturbations of the metric) which makes the search for the B-mode polarization crucially linked to a probe of the inflationary scenario. The amplitude of this background is determined by the energy scale of inflation, which can widely vary among different inflationary models. From this point of view, future satellite missions, such as *Planck*, will have enough sensitivity to either detect or constrain the B-mode CMB polarization predicted by the simplest inflationary models. A lot of experimental and theoretical issues are currently under investigations in order to assess the feasibility of a detection of the primordial

---

\*Electronic address: nicola.bartolo@pd.infn.it

†Electronic address: sabino.matarrese@pd.infn.it

‡Electronic address: mollerach@cab.cnea.gov.ar

§Electronic address: antonio.riotto@pd.infn.it

gravity waves via the B-mode polarization. The main problematic aspects come from foregrounds [3], instrumental noise [4], and the gravitational lensing on the CMB by the matter distribution. These effects might actually mask the signal due to primordial tensor modes. As far as the lensing “contamination” is concerned, this is a non-linear effect that implies the transformation of E-mode into B-mode polarization [5] and it has been pointed out that the inflationary gravitational-wave background can only be detected by CMB polarization measurements if the tensor to scalar ratio  $r \geq 10^{-4}$ , which corresponds to an energy scale of inflation larger than  $3 \times 10^{15}$  GeV [6, 7, 8, 9]. However, a better technique to *clean* polarization maps from the lensing effect has been proposed, which would allow tensor-to-scalar ratios as low as  $10^{-6}$ , or even smaller, to be probed [10, 11].

In this paper we point out the existence of a new contribution to the B-mode polarization. It is due to the observation made in Ref. [12] that at second-order in perturbation theory the quadrupole moment of the photons is not suppressed by the inverse of the optical depth in the tight coupling limit, as it happens at first order in perturbation theory. The anisotropic Thomson scattering of the photon quadrupole leads to a polarization of the CMB. The study of the temperature anisotropies at second-order have been analyzed in several works [13] and recently Refs. [12, 14] studied the evolution of the temperature anisotropies in the tight coupling regime through the full derivation of the Boltzmann equations for the photon, baryon and cold dark matter components. In Ref. [15] it has been shown how B-mode polarization is produced during the last scattering and reionization epochs by the non-linear evolution of cosmological perturbations which generates unavoidably vector and tensor modes, starting from the primordial (scalar) density fluctuations.

In our case it is the non-suppression of the second-order quadrupole in the tight coupling regime that makes the contribution to the B-mode polarization considered here of a different origin than that of Ref. [15]. Our findings show that the corresponding level of the B-mode is larger and also that it overcomes by two orders of magnitude the analogous signal generated during the reionization epoch [16]. It turns out that such a novel contribution to the CMB B-mode constitutes a background to the detection of the primordial gravitational waves effect if the tensor to scalar ratio is smaller than a few  $\times 10^{-5}$ . The magnitude of the effect is smaller than the contamination produced by the conversion of polarization of type E into type B, by weak gravitational lensing. However the lensing signal can be cleaned, making the secondary modes discussed here the actual background contaminating the detection of small amplitude primordial gravitational waves.

These secondary effects always exist and their amplitude has a one-to-one relation with the level of density perturbations, which is severely constrained by both CMB anisotropy measurements and Large-Scale Structure observations. Therefore, their properties are largely inflation model-independent, contrary to primary tensor modes whose amplitude is not only model-dependent, but is well-known to be suppressed in some cases, like e.g. in the so-called curvaton model for the generation of curvature perturbations [17].

The plan of the paper is as follows. In Sec. II we recall the origin of the quadrupole moment in the tight coupling limit at second-order in perturbation theory. In Sec. III the contribution to the power spectrum of the B-mode polarization from the second-order quadrupole is computed and finally in Sec. IV we compare the results to the contribution to the B-mode from the inflationary tensor modes and the gravitational lensing and with other secondary effects as well.

## II. SECOND-ORDER QUADRUPOLE MOMENT OF PHOTONS AND THE TIGHT COUPLING LIMIT

In this section we recall the starting point of our paper: at second (and higher) order in the perturbations the quadrupole moment of the photon distribution cannot be neglected even in the limit of tight coupling. At linear order the quadrupole and higher order moments of the photons, in the tight coupling limit, are negligible with respect to the first two moments (the energy density and velocity) because they turn out to be suppressed by increasing powers of the inverse optical depth  $\tau^{-1}$ . However in the non-linear regime there is a source for the quadrupole made up of linear-velocity squared terms which do not vanish in the tight coupling limit. Such a result has been obtained in details in Ref. [12], and in the following we limit ourselves to give some definitions and summarize the main steps necessary to achieve it.

The quadrupole moment of the photon distribution is defined as

$$\Pi_{\gamma}^{ij} = \int \frac{d\Omega}{4\pi} \left( n^i n^j - \frac{1}{3} \delta^{ij} \right) \Delta, \quad (1)$$

where  $\Delta = \Delta^{(1)} + \Delta^{(2)}/2$  are the photon temperature anisotropies split into a first and a second-order part. They are given by

$$\Delta(x^i, n^i, \eta) = \frac{\int dp p^3 \delta f}{\int dp p^3 f^{(0)}}, \quad (2)$$

with  $\eta$  the conformal time, and  $f^{(0)}$  the zeroth-order (Bose-Einstein) value around which the distribution function of photons is expanded,  $f = f^{(0)} + \delta f = f^{(0)} + f^{(1)} + f^{(2)}/2$ . Eq. (2) represents the photon fractional energy perturbation (in a given direction), which is the integral of the photon distribution function perturbation  $\delta f$  over the photon momentum magnitude  $p$  ( $p^i = pn^i$ ). The angular dependence of the photon anisotropies  $\Delta$  can be expanded as

$$\Delta(\mathbf{x}, \mathbf{n}) = \sum_{\ell} \sum_{m=-\ell}^{\ell} \Delta_{\ell m}(\mathbf{x}) (-i)^{\ell} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}(\mathbf{n}), \quad (3)$$

with

$$\Delta_{\ell m} = (-i)^{-\ell} \sqrt{\frac{2\ell+1}{4\pi}} \int d\Omega \Delta Y_{\ell m}^*(\mathbf{n}). \quad (4)$$

The Boltzmann equations up to second-order for the photon temperature anisotropies  $\Delta$  (and the hierarchy equations for  $\Delta_{\ell m}$ ) have been obtained in Ref. [14], together with the second-order Boltzmann equations for the baryons and cold dark matter. One can integrate the R.H.S. of the Boltzmann equation for  $\Delta^{(2)}$  over  $d\Omega_{\mathbf{n}}(n^i n^j - \delta^{ij}/3)/4\pi$  and set it to be vanishing in the limit of tight coupling. Then the R.H.S. becomes [12]

$$\frac{\dot{\tau}}{2} \left[ -\Pi_{\gamma}^{ij} + \frac{1}{10} \Pi_{\gamma}^{ij} + (\delta_b + \Psi) \left( -2\Pi_{\gamma}^{ij} - \frac{1}{25} \Delta_{20} (\hat{v}^i \hat{v}^j - \frac{1}{3} \delta^{ij}) \right) + \frac{12}{5} \left( v^i v^j - \frac{1}{3} \delta^{ij} v^2 \right) \right], \quad (5)$$

where  $v^i$  is the photon-baryon velocity,  $\delta_b$  the baryon density contrast,  $\Psi$  the gravitational potential, and the differential optical depth  $\dot{\tau} = \bar{n}_e \sigma_T a$  sets the collision rate in conformal time, with  $\bar{n}_e$  the mean free electron density and  $\sigma_T$  the Thomson cross section.<sup>1</sup> Therefore, in the limit of tight coupling, when the interaction rate is very high, the second-order quadrupole moment is given by

$$\Pi_{\gamma}^{ij} \simeq \frac{8}{3} \left( v^i v^j - \frac{1}{3} \delta^{ij} v^2 \right), \quad (7)$$

as it follows by requiring that Eq. (5) vanishes (the term multiplying  $(\delta_b + \Psi)$  goes to zero in the tight coupling limit since it just comes from the first-order collision term). At linear order one would simply get the term  $(-9\dot{\tau}\Pi_{\gamma}^{ij}/10)$  implying that, in the limit of a high scattering rate  $\dot{\tau}$ ,  $\Pi_{\gamma}^{ij}$  goes to zero. However at second-order the quadrupole is not suppressed in the tight coupling limit because it turns out to be sourced by the linear-velocity squared.

Indeed the result of Eq. (7) is not surprising and had to be expected. First, the limit of tight coupling corresponds to treat the photons in the fluid approximation, and, at second-order, the energy momentum tensor of a fluid is exactly characterized by an anisotropic stress given by Eq. (7) (see, for example, Ref. [20]) Moreover it is also known that, besides the usual dipole contribution, beyond linear order a quadratic Doppler effect gives rise to a quadrupole anisotropy (see Refs. [16, 21, 22]).

### III. CMB POLARIZATION

Since the polarization of the CMB is generated by the scattering of the quadrupole anisotropies by free electrons, armed with Eq. (7), we will explore the effects of such a contribution to the quadrupole on the CMB polarization.<sup>2</sup> Notice that the resulting polarization power spectrum has a different origin from that computed in Ref. [15]. Ref. [15] shows how second-order vector and tensor perturbations can produce CMB polarization through the decoupling phase. Once the vector and the tensor perturbations are obtained at second-order as a non-linear combination of first-order

<sup>1</sup> The perturbed line element around a spatially flat FRW background is taken in the so-called Poisson gauge [18, 19], which eliminates one scalar degree of freedom from the  $g_{0i}$  component and one scalar and two vector degrees of freedom from  $g_{ij}$ , thus reducing at linear order to the Newtonian gauge

$$ds^2 = a^2(\eta) \left\{ -(1 + 2\Psi) d\eta^2 - 2V_i d\eta dx^i + [(1 - 2\Phi) \delta_{ij} + 2H_{ij}] dx^i dx^j \right\}. \quad (6)$$

Each perturbation can be decomposed into a linear and a second-order term, e.g.  $\Psi = \Psi^{(1)} + \Psi^{(2)}/2$ .  $V_i$  contains scalar and vector (divergence-free) perturbation modes, while  $H_{ij}$  is a tensor mode (i.e. traceless and divergence-free,  $\partial^i H_{ij} = H^i_i = 0$ ).

<sup>2</sup> The effects of the quadrupole moment (7) on the CMB temperature anisotropies have already been studied in Ref. [12].

scalar perturbations, the induced power spectra of the E- and B-mode polarization are computed using the standard techniques developed at linear order [23]: a quadrupole is generated by the vector and tensor perturbations only beyond the leading order in the tight coupling expansion (otherwise it is suppressed) and to evaluate it one has to carefully account for the epoch around decoupling. The result is thus proportional to  $\Delta\eta_*$ , the width of the visibility function (i.e, the thickness of the last scattering surface). The case of Eq. (7) is completely different in that it is a quadrupole that is indeed generated in the tight coupling limit, and the corresponding power spectra of CMB polarization will be computed following different approximations as well.

### A. Polarization angular power spectra

In order to study the generation of polarization anisotropies we will employ the formalism of the total angular momentum method [24, 25] which has the advantage of putting scalar, vector and tensor perturbations on an equal footing. This is the most convenient approach for our calculations, and we are going to extensively use the results of Ref. [24]. The temperature and polarization fluctuations are expanded in normal modes that take into account the dependence on both the angular direction of photon propagation  $\mathbf{n}$  and the spatial position  $\mathbf{x}$ ,  ${}_sG_\ell^m(\mathbf{x}, \mathbf{n})$ <sup>3</sup>

$$\begin{aligned}\Theta(\mathbf{x}, \mathbf{n}, \eta) &= \int \frac{d^3k}{(2\pi)^3} \sum_\ell \sum_{m=-2}^2 \Theta_\ell^{(m)} {}_0G_\ell^m, \\ (Q \pm iU)(\mathbf{x}, \mathbf{n}, \eta) &= \int \frac{d^3k}{(2\pi)^3} \sum_\ell \sum_{m=-2}^2 (E_\ell^{(m)} \pm iB_\ell^{(m)}) {}_{\pm 2}G_\ell^m,\end{aligned}\tag{8}$$

with spin  $s = 0$  describing the temperature fluctuation and  $s = \pm 2$  describing the polarization tensor and  $m = 0, \pm 1, \pm 2$  denoting scalar, vector and tensor perturbations, respectively.  $E_\ell^{(m)}$  and  $B_\ell^{(m)}$  are the angular moments of the electric and magnetic polarization components and

$${}_sG_\ell^m(\mathbf{x}, \mathbf{n}) = (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} [{}_sY_\ell^m(\mathbf{n})] \exp(i\mathbf{k} \cdot \mathbf{x}).\tag{9}$$

Before keeping on recalling the main results of the total angular momentum method, a comment is in order here. The Boltzmann equations for the temperature and polarization anisotropies we are going to write are derived within linear perturbation theory. Then, given the second-order quadrupole (7), our approach will be that of evolving it linearly into Eqs. (17). In fact this is the correct procedure if we wish to isolate its contribution to the CMB polarization at second-order, distinguishing it from other unavoidable corrections to the equations (17) coming in the form of first-order squared terms. In a similar way, the variable  $\Delta^{(2)}(\mathbf{x}, \mathbf{n}, \eta)$  will be taken to be  $4\Theta(\mathbf{x}, \mathbf{n}, \eta)$ , a relation holding at the linear level (see Refs. [12, 26]).

The Boltzmann equation describing the time evolution of the radiation distribution under gravitation and scattering processes can be written as a set of evolution equations for the angular moments of the temperature,  $\Theta_\ell^{(m)}$  (for  $\ell \geq m$ ), and both polarization types,  $E_\ell^{(m)}$  and  $B_\ell^{(m)}$  (for  $\ell \geq 2$  and  $m \geq 0$ ),

$$\dot{\Theta}_\ell^{(m)} = k \left[ \frac{{}_0\kappa_\ell^m}{(2\ell-1)} \Theta_{\ell-1}^{(m)} - \frac{{}_0\kappa_{\ell+1}^m}{(2\ell+3)} \Theta_{\ell+1}^{(m)} \right] - \dot{\tau} \Theta_\ell^{(m)} + S_\ell^{(m)},\tag{10}$$

$$\dot{E}_\ell^{(m)} = k \left[ \frac{2\kappa_\ell^m}{(2\ell-1)} E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell+1)} B_\ell^{(m)} - \frac{2\kappa_{\ell+1}^m}{(2\ell+3)} E_{\ell+1}^{(m)} \right] - \dot{\tau} [E_\ell^{(m)} + \sqrt{6} P^{(m)} \delta_{\ell,2}],\tag{11}$$

$$\dot{B}_\ell^{(m)} = k \left[ \frac{2\kappa_\ell^m}{(2\ell-1)} B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell+1)} E_\ell^{(m)} - \frac{2\kappa_{\ell+1}^m}{(2\ell+3)} B_{\ell+1}^{(m)} \right] - \dot{\tau} B_\ell^{(m)},\tag{12}$$

where a dot stands for a derivative with respect to the conformal time  $\eta$  and the coupling coefficients are

$${}_s\kappa_\ell^m = \sqrt{\frac{(\ell^2 - m^2)(\ell^2 - s^2)}{\ell^2}}.\tag{13}$$

---

<sup>3</sup> In order to avoid confusion with the notations of Ref. [24], from now on, if not explicitly written, we are going to drop the superscript denoting the order of the perturbations.

The fluctuation sources are given by

$$\begin{aligned} S_0^{(0)} &= \dot{\tau} \Theta_0^{(0)} - \dot{\Phi}, \quad S_1^{(0)} = \dot{\tau} v_B^{(0)} + k\Psi, \quad S_2^{(0)} = \dot{\tau} P^{(0)}, \\ S_1^{(1)} &= \dot{\tau} v_B^{(1)} + \dot{V}, \quad S_2^{(1)} = \dot{\tau} P^{(1)}, \\ S_2^{(2)} &= \dot{\tau} P^{(2)} - \dot{H}, \end{aligned} \quad (14)$$

with

$$P^{(m)} = \frac{1}{10} \left[ \Theta_2^{(m)} - \sqrt{6} E_2^{(m)} \right]. \quad (15)$$

The modes with  $m = -|m|$  satisfy the same equations with  $B_\ell^{(-|m|)} = -B_\ell^{(|m|)}$  and all the other quantities unchanged. In Eq. (14)  $v_B$  is the baryon velocity perturbation which in the tight coupling regime corresponds to the photon-baryon velocity  $v$ .

These equations can be formally integrated, leading to simple expressions in terms of an integral along the line-of-sight [24]. The temperature fluctuations are given by

$$\frac{\Theta_\ell^{(m)}(\eta, k)}{2\ell + 1} = \int_0^\eta d\eta' e^{-\tau} \sum_{\ell'} S_{\ell'}^{(m)}(\eta') j_\ell^{(\ell' m)}(k(\eta - \eta')), \quad (16)$$

where  $j_\ell^{(\ell' m)}$  are given in Ref. [24]. For the polarization, we have

$$\begin{aligned} \frac{E_\ell^{(m)}(\eta_0, k)}{2\ell + 1} &= -\sqrt{6} \int_0^{\eta_0} d\eta \dot{\tau} e^{-\tau} P^{(m)}(\eta) \epsilon_\ell^{(m)}(k(\eta_0 - \eta)), \\ \frac{B_\ell^{(m)}(\eta_0, k)}{2\ell + 1} &= -\sqrt{6} \int_0^{\eta_0} d\eta \dot{\tau} e^{-\tau} P^{(m)}(\eta) \beta_\ell^{(m)}(k(\eta_0 - \eta)), \end{aligned} \quad (17)$$

where the radial functions read

$$\begin{aligned} \epsilon_\ell^{(\pm 1)}(x) &= \frac{1}{2} \sqrt{(\ell - 1)(\ell + 2)} \left[ \frac{j_\ell(x)}{x^2} + \frac{j'_\ell(x)}{x} \right], \\ \epsilon_\ell^{(\pm 2)}(x) &= \frac{1}{4} \left[ -j_\ell(x) + j''_\ell(x) + 2 \frac{j_\ell(x)}{x^2} + 4 \frac{j'_\ell(x)}{x} \right], \end{aligned} \quad (18)$$

$$\begin{aligned} \beta_\ell^{(+1)}(x) &= -\beta_\ell^{(-1)}(x) = \frac{1}{2} \sqrt{(\ell - 1)(\ell + 2)} \frac{j_\ell(x)}{x}, \\ \beta_\ell^{(+2)}(x) &= -\beta_\ell^{(-2)}(x) = \frac{1}{2} \left[ j'_\ell(x) + 2 \frac{j_\ell(x)}{x} \right]. \end{aligned} \quad (19)$$

The differential optical depth  $\dot{\tau} = \bar{n}_e \sigma_T a$  sets the collision rate in conformal time, with  $\bar{n}_e$  the mean free electron density and  $\sigma_T$  the Thomson cross section and  $\tau(\eta_0, \eta) \equiv \int_\eta^{\eta_0} \dot{\tau}(\eta') d\eta'$  the optical depth between  $\eta$  and the present time. The combination  $g(\eta) = \dot{\tau} e^{-\tau}$  is the *visibility function* and it expresses the probability that a photon last scattered between  $d\eta$  of  $\eta$  (with  $\int d\eta g(\eta) = 1$ ) and hence it is sharply peaked at the last scattering epoch. In early reionization models, a second peak is also present at more recent times.

Scalar modes do not contribute to B-polarization, thus  $B_\ell^{(0)} = 0$ . We are interested in the contribution to the angular power-spectrum for the E and B modes arising from vector ( $m = 1$ ) and tensor ( $m = 2$ ) perturbations,

$$\begin{aligned} C_\ell^{(E)} &= \frac{2}{\pi} \int \frac{dk}{k} \sum_{m=-2}^2 k^3 \frac{|E_\ell^{(m)}(\eta_0, k)|^2}{(2\ell + 1)^2}, \\ C_\ell^{(B)} &= \frac{2}{\pi} \int \frac{dk}{k} \sum_{m=-2}^2 k^3 \frac{|B_\ell^{(m)}(\eta_0, k)|^2}{(2\ell + 1)^2}. \end{aligned} \quad (20)$$

In the following we will focus only on the B-mode of CMB polarization. This is probably the most interesting case when confronting the signal generated by the primordial gravitational waves with that coming from second-order perturbations. Both are expected to be very low, but for different reasons: the former because of the low content of primordial gravitational waves, the latter because it is a second-order effect. Then, as first pointed out in Ref. [15], there could be the possibility that for a low value of primordial tensor to scalar ratio  $r$ , this kind of secondary signal could constitute an additional barrier to our ability in detecting the signature of the inflationary gravitational waves.

### B. B-mode polarization from the tight coupling quadrupole

The first step is to compute the polarization source term  $P^{(m)}$  in Eq. (17). In the tight coupling limit we deduce from Eq. (11) that  $\Theta_2^{(m)} = -4E_2^{(m)}/\sqrt{6}$ , where we have used Eq. (15). Plugging this result back into Eq. (15) yields

$$P^{(m)} = \frac{\Theta_2^{(m)}}{4}. \quad (21)$$

The B-mode polarization field is then obtained from Eq. (17) exploiting the fact that the visibility function is sharply peaked around the epoch of decoupling  $\eta_*$ . We thus evaluate  $P^{(m)}(\eta)$  and  $\beta_\ell^{(1)}(k(\eta_0 - \eta))$  at  $\eta_*$  taking them out of the integrals

$$\frac{B_\ell^{(m)}(k, \eta_0)}{2\ell + 1} = \frac{\sqrt{6}}{16} \Delta_{2m}(k, \eta_*) \beta_\ell^{(1)}(k(\eta_0 - \eta_*)), \quad (22)$$

where we have used  $\Delta(\mathbf{x}, \mathbf{n}, \eta) = 4\Theta(\mathbf{x}, \mathbf{n}, \eta)$ . Notice the different approximations used in deriving Eq. (22) with respect to the procedure of Ref. [15]; in particular we do not need to go beyond the leading order in the tight coupling approximation, and this allows an easier evaluation of the time integral (17).

The quantities  $\Delta_{2m}$  are related to the quadrupole defined in Eq. (1). If in Fourier space for a given  $\mathbf{k}$  we choose  $\mathbf{e}_3 = \hat{\mathbf{k}}$  and  $\mathbf{e}_1$  and  $\mathbf{e}_2$  forming an orthonormal basis, it is easy to verify that

$$\Delta_{20} = -\frac{15}{2} e_{3i} e_{3j} \Pi_\gamma^{ij}, \quad (23)$$

$$\Delta_{2\pm 1} = \pm 5\sqrt{3} e_{3i} \frac{(\mathbf{e}_1 \mp i\mathbf{e}_2)_j}{\sqrt{2}} \Pi_\gamma^{ij}, \quad (24)$$

$$\Delta_{2\pm 2} = -5\sqrt{\frac{3}{2}} \frac{(\mathbf{e}_1 \mp i\mathbf{e}_2)_i}{\sqrt{2}} \frac{(\mathbf{e}_1 \mp i\mathbf{e}_2)_j}{\sqrt{2}} \Pi_\gamma^{ij}, \quad (25)$$

where one has to use Eq.(4) and the explicit expression of the spherical harmonics in the chosen coordinate system. Being interested in the B-mode polarization in the following we will make use only of the last two expressions, for the multipoles  $\pm 1$  and  $\pm 2$ , corresponding to the vector and tensor parts of the quadrupole moment.

The (first-order) velocities appearing in (7) are those of the tightly coupled fluid of photons and baryons, and we will use the analytical solutions found in Ref. [12] with

$$v_\gamma^i = -i \frac{k^i}{k} \frac{9}{10} \Phi_{\mathbf{k}}^{(1)}(0) \sin(kc_s\eta) c_s \quad (k \ll k_{eq}), \quad (26)$$

and

$$v_\gamma^i = -i \frac{k^i}{k} \frac{9}{2} \Phi_{\mathbf{k}}^{(1)}(0) \sin(kc_s\eta) c_s \quad (k \gg k_{eq}), \quad (27)$$

for modes entering the horizon before or after the equality epoch, respectively. Here  $\Phi_{\mathbf{k}}^{(1)}(0)$  is the primordial linear gravitational potential on superhorizon scales deep in the radiation dominated epoch, while  $c_s = (3(1 + R_*))^{-1/2}$  is the sound speed of the photon-baryon fluid, with  $R_* = 3\rho_b/(4\rho_\gamma)$  the ratio of the baryon to photon energy density at the decoupling epoch.

Using Eq. (26) and (27) in Eqs. (24) and (25) the temperature quadrupole anisotropies can be written as a convolution

$$\Delta_{2m}(\mathbf{k}, \eta_*) = \int \frac{d^3 k_1}{(2\pi)^3} f_m(\mathbf{k}_1, \mathbf{k} - \mathbf{k}_1) \Phi_{\mathbf{k}_1}^{(1)}(0) \Phi_{\mathbf{k} - \mathbf{k}_1}^{(1)}(0), \quad (28)$$

where the kernel for the vector part,  $m = \pm 1$ , reads

$$f_{\pm 1}(\mathbf{k}_1, \mathbf{k} - \mathbf{k}_1) \hat{\mathbf{k}}_1^j \hat{\mathbf{k}}_2^i = \mp \frac{20}{\sqrt{6}} N c_s^2 \sin(k_1 c_s \eta_*) \sin(|\mathbf{k} - \mathbf{k}_1| c_s \eta_*) e^{\mp i\phi_{\mathbf{k}_1}} \sin \theta_{\mathbf{k}_1} \left[ \frac{k}{|\mathbf{k} - \mathbf{k}_1|} - 2 \cos \theta_{\mathbf{k}_1} \frac{k_1}{|\mathbf{k} - \mathbf{k}_1|} \right], \quad (29)$$

a vector normalized to unity, e.g. and that for the tensor part,  $m = \pm 2$ , is given by

$$f_{\pm 2}(\mathbf{k}_1, \mathbf{k} - \mathbf{k}_1) = -\frac{20}{\sqrt{6}} N c_s^2 \sin(k_1 c_s \eta_*) \sin(|\mathbf{k} - \mathbf{k}_1| c_s \eta_*) e^{\mp 2i\phi_{\mathbf{k}_1}} \sin \theta_{\mathbf{k}_1} \frac{k_1}{|\mathbf{k} - \mathbf{k}_1|}. \quad (30)$$

To compute Eqs. (29) and (30) one makes use of the orthonormality of the basis  $\mathbf{e}_3 = \hat{\mathbf{k}}$ ,  $\mathbf{e}_1$  and  $\mathbf{e}_2$ ;  $\phi_{\mathbf{k}_1}$  and  $\theta_{\mathbf{k}_1}$  are the azimuthal and polar angles of  $\mathbf{k}_1$  in such a coordinate system. The coefficient  $N$  comes from the prefactors appearing in the expressions for the linear velocities (26) and (27), with  $N = (9/10)^2$  if both  $k_1$  and  $|\mathbf{k} - \mathbf{k}_1|$  are less than  $k_{eq}$ ,  $N = (9/2)^2$  if both the wavenumbers are greater than  $k_{eq}$ , while in the mixed cases  $N = (9/10) \times (9/2)$ .

The power spectrum of the quadrupole anisotropies is defined by

$$\langle \Delta_{2m}(\mathbf{k}) \Delta_{2m}^*(\mathbf{p}) \rangle = (2\pi)^3 P_{\Delta_{2m}}(k) \delta^{(3)}(\mathbf{k} - \mathbf{p}), \quad (31)$$

and, using the general expression (28) it turns out to be

$$P_{\Delta_{2m}}(k) = 2 \int \frac{d^3 k_1}{(2\pi)^3} |f_m(\mathbf{k}_1, \mathbf{k} - \mathbf{k}_1)|^2 P_\Phi(k_1) P_\Phi(|\mathbf{k} - \mathbf{k}_1|), \quad (32)$$

where  $P_\Phi(k_1)$  is the power spectrum of the primordial gravitational potential. The power spectrum of the quadrupole anisotropies enters in the quantity we are interested in, namely the contribution to the angular power spectrum of the B-mode from tensor and vector perturbations as given by Eq. (20). Inserting Eq. (22) we find

$$C_\ell^{(B)(m)} = \frac{2}{\pi} \int dk k^2 \frac{|B_\ell^{(m)}(k, \eta_0)|^2}{(2\ell + 1)^2} = \frac{3}{64\pi} \int dk k^2 P_{\Delta_{2m}}(k, \eta_*) \left[ \beta_\ell^{(m)}(k(\eta_0 - \eta_*)) \right]^2. \quad (33)$$

Let us now write down explicitly the power spectra (32) using Eqs. (29)-(30). For the vector perturbation mode we find

$$\begin{aligned} P_{\Delta_{2\pm 1}}(k, \eta_*) &= \frac{100}{3} \frac{c_s^4}{\pi^2} \int_{-1}^{+1} d(\cos \theta_{\mathbf{k}_1}) (1 - \cos^2 \theta_{\mathbf{k}_1}) \int_0^\infty dk_1 N^2 k_1^2 \sin^2(k_1 c_s \eta_*) \sin^2(|\mathbf{k} - \mathbf{k}_1| c_s \eta_*) \\ &\times \left[ \frac{k}{|\mathbf{k} - \mathbf{k}_1|} - 2 \cos \theta_{\mathbf{k}_1} \frac{k_1}{|\mathbf{k} - \mathbf{k}_1|} \right]^2 P_\Phi(k_1) P_\Phi \left( \sqrt{k^2 + k_1^2 - 2kk_1 \cos \theta_{\mathbf{k}_1}} \right), \end{aligned} \quad (34)$$

while for the tensor mode

$$\begin{aligned} P_{\Delta_{2\pm 2}}(k, \eta_*) &= \frac{100}{3} \frac{c_s^4}{\pi^2} \int_{-1}^{+1} d(\cos \theta_{\mathbf{k}_1}) (1 - \cos^2 \theta_{\mathbf{k}_1}) \int_0^\infty dk_1 N^2 \frac{k_1^4}{|\mathbf{k} - \mathbf{k}_1|^2} \sin^2(k_1 c_s \eta_*) \sin^2(|\mathbf{k} - \mathbf{k}_1| c_s \eta_*) \\ &\times P_\Phi(k_1) P_\Phi \left( \sqrt{k^2 + k_1^2 - 2kk_1 \cos \theta_{\mathbf{k}_1}} \right). \end{aligned} \quad (35)$$

An efficient way to handle the above integrals over the wavenumber  $\mathbf{k}_1$  is to introduce the variables

$$x = \frac{|\mathbf{k} - \mathbf{k}_1|}{k}, \quad y = \frac{k_1}{k}. \quad (36)$$

With such a substitution we arrive at a compact expression for the angular power spectra of the B-mode polarization

$$\begin{aligned} C_\ell^{(B)(\pm 1)} &= \frac{25}{144} \frac{1}{\pi^3} \int dk k^5 \int_0^\infty dy y \int_{|1-y|}^{|1+y|} \frac{dx}{x} \left[ 1 - \left( \frac{1+y^2-x^2}{2y} \right)^2 \right] \\ &\times (y^2 - x^2)^2 N^2 \sin^2(k c_s \eta_* y) \sin^2(k c_s \eta_* x) P_\Phi(ky) P_\Phi(kx) \left[ \beta_\ell^{(\pm 1)}(k(\eta_0 - \eta_*)) \right]^2, \end{aligned} \quad (37)$$

$$\begin{aligned} C_\ell^{(B)(\pm 2)} &= \frac{25}{144} \frac{1}{\pi^3} \int dk k^5 \int_0^\infty dy y^3 \int_{|1-y|}^{|1+y|} \frac{dx}{x} \left[ 1 - \left( \frac{1+y^2-x^2}{2y} \right)^2 \right]^2 \\ &\times N^2 \sin^2(k c_s \eta_* y) \sin^2(k c_s \eta_* x) P_\Phi(ky) P_\Phi(kx) \left[ \beta_\ell^{(\pm 2)}(k(\eta_0 - \eta_*)) \right]^2. \end{aligned} \quad (38)$$

We remind the reader that  $N = (9/10)^2$  if both  $k_1 = ky$  and  $|\mathbf{k} - \mathbf{k}_1| = kx$  are less than  $k_{eq}$ ,  $N = (9/2)^2$  if both the wavenumbers are greater than  $k_{eq}$ , while in the mixed cases  $N = (9/10) \times (9/2)$ . The power spectrum of the

primordial gravitational potential can be written as  $P_\Phi = P_{0\Phi} k^{-3} (k/k_0)^{n_s-1}$ , where  $n_s$  is the scalar spectral index and the amplitude is related to that of the comoving curvature perturbation spectrum,  $\Delta_{\mathcal{R}}^2(k) = \Delta_{\mathcal{R}}^2(k_0) (k/k_0)^{n_s-1}$ , by  $P_{0\Phi} = (4/9)2\pi^2 \Delta_{\mathcal{R}}^2(k_0)$ . Therefore Eqs. (37) and (38) become

$$C_\ell^{(B)(\pm 1)} = P_{0\Phi}^2 \frac{25}{144} \frac{1}{\pi^3} \int \frac{dk}{k} \left( \frac{k}{k_0} \right)^{2(n_s-1)} \int_0^\infty dy y^{n_s-3} \int_{|1-y|}^{|1+y|} dx x^{n_s-5} (y^2 - x^2)^2 \\ \times \left[ 1 - \left( \frac{1+y^2-x^2}{2y} \right)^2 \right]^2 N^2 \sin^2(kc_s \eta_* y) \sin^2(kc_s \eta_* x) \left[ \beta_\ell^{(\pm 1)}(k(\eta_0 - \eta_*)) \right]^2, \quad (39)$$

and

$$C_\ell^{(B)(\pm 2)} = P_{0\Phi}^2 \frac{25}{144} \frac{1}{\pi^3} \int \frac{dk}{k} \left( \frac{k}{k_0} \right)^{2(n_s-1)} \int_0^\infty dy y^{n_s-1} \int_{|1-y|}^{|1+y|} dx x^{n_s-5} \\ \times \left[ 1 - \left( \frac{1+y^2-x^2}{2y} \right)^2 \right]^2 N^2 \sin^2(kc_s \eta_* y) \sin^2(kc_s \eta_* x) \left[ \beta_\ell^{(\pm 2)}(k(\eta_0 - \eta_*)) \right]^2. \quad (40)$$

To perform the integration in  $k$  it is convenient to use the WKB approximated expressions

$$\langle (\beta_\ell^{(2)})^2(x) \rangle \simeq \frac{1}{8} \left( \frac{\sqrt{x^2 - \ell^2}}{x^3} + \frac{4}{x^3 \sqrt{x^2 - \ell^2}} \right) \quad (41) \\ \langle (\epsilon_\ell^{(2)})^2(x) \rangle \simeq \frac{1}{8} \left[ \left( 1 - \frac{1 + \ell(\ell+1)/2}{x^2} \right)^2 \frac{1}{x \sqrt{x^2 - \ell^2}} + \frac{\sqrt{x^2 - \ell^2}}{x^5} \right],$$

for  $x > \ell$ , and 0 for  $x < \ell$ .

Notice that up to now we have not accounted for the diffusion damping effects in writing these integrals. This can be done by replacing the sines with

$$\sin^2(kc_s \eta_* x) \rightarrow \sin^2(kc_s \eta_* x) e^{-2 \frac{k^2 x^2}{k_D^2}}, \quad (42)$$

$$\sin^2(kc_s \eta_* y) \rightarrow \sin^2(kc_s \eta_* y) e^{-2 \frac{k^2 y^2}{k_D^2}}. \quad (43)$$

where  $k_D$  is evaluated at the decoupling epoch and defines the damping scale [27].

#### IV. DISCUSSION AND CONCLUSIONS

We show in Figure 1 the results for the vector and tensor contribution to the B-mode polarization multipoles for a  $\Lambda$ CDM model with scalar spectral index  $n = 0.95$  and normalization  $\Delta_{\mathcal{R}}^2(k_0) = 2.3 \times 10^{-9}$  at  $k_0 = 0.002/\text{Mpc}$  (the damping scale has been chosen so that  $k_D \eta_0 \simeq 1600$ ). The tensor modes provide the dominant contribution for  $\ell < 100$ , while the vector modes contribution dominates at larger values of  $\ell$ . Notice that the structure of the peaks of the second-order effect is richer than the usual (linear) polarization power spectra because the quadrupole signal is modulated as a velocity squared giving rise to a  $\sin^2(kc_s \eta_*)$  dependence, which appears with two different periods in the convolution products (34)-(35).

Fig. 1 reveals that the second-order quadrupole B-mode polarization computed here is larger for  $\ell > 10$  than that due to the second-order vector and tensor modes, which are produced by the non-linear evolution of primordial scalar perturbations, considered in Ref. [15]. One could also ask what is the analogous signature on the B-mode CMB polarization induced by the second-order quadrupole moment of the photons during the reionization epoch. In this case the magnitude of the effect has been already computed in Ref. [16]. A comparison of our results in Fig. 1 to those shown in Fig. 9 of Ref. [16] shows that the contribution of the second-order quadrupole from recombination overcomes the analogous signal generated during the reionization epoch by two orders of magnitude in the relevant range  $100 \leq \ell \leq 1000$ .

It is also interesting to compare this second-order quadrupole contribution to the B-mode polarization with that expected from the primordial gravitational waves background generated during inflation. This depends of course on



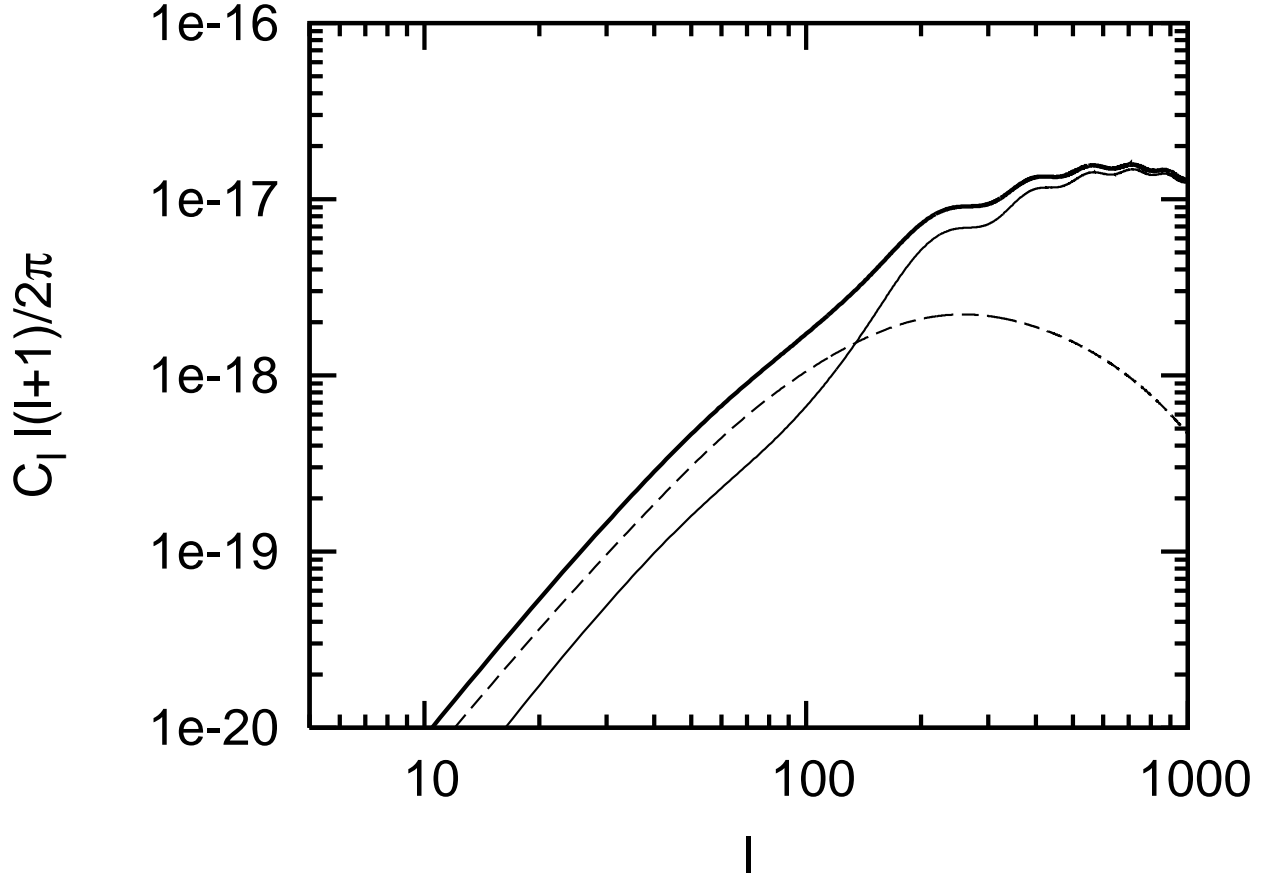


FIG. 1: Angular power-spectrum of B polarization originated at recombination by the second-order quadrupole. The thin solid line represents the vector modes contribution and the dashed line the tensor modes. The thick solid line correspond to the total contribution.

the model of inflation considered and its amplitude is parametrized by the ratio of the tensor ( $h$ ) to scalar power spectrum amplitudes  $r = \Delta_h^2(k_0)/\Delta_{\mathcal{R}}^2(k_0)$  (accordingly to the definition used in the WMAP analysis [28, 29]). In Figure 2 we show the primordial gravitational wave B polarization multipoles for a model with  $r = 10^{-5}$  (computed with the help of the CMBFAST code [30]), along with the total (vector plus tensor) contribution from the second-order quadrupole at recombination. We have considered for the  $\Lambda$ CDM model the cosmological parameters  $\Omega_\Lambda = 0.732$ ,  $\Omega_m = 0.268$ ,  $\Omega_b h^2 = 0.0218$ ,  $h = 0.7$ ,  $\tau = 0.073$  [29]. The contribution from the primordial gravitational waves scales linearly with  $r$ , thus we see that for values of  $r$  smaller than a few  $\times 10^{-5}$  the second-order quadrupole contribution becomes dominant for  $\ell > 10$ , limiting our ability to detect the primordial gravitational-wave background. The peak from the primordial gravitational waves at the lowest multipoles, arising from the reionization epoch, could be affected by the contribution studied here, only for extremely low values of the scalar to tensor ratio ( $r < 10^{-8}$ ).

The main background for the detection of the primordial gravitational waves through their B-mode polarization signal is however due to the weak gravitational lensing conversion of the dominant E modes polarization to B modes [5]. Fortunately, this effect can be, at least partially, accounted for through reconstruction of the gravitational lensing potential by the correlations of B-polarization between large and small angular scales, which primordial gravitational waves do not produce. “Cleaning” of the gravitational lensing signature may achieve factors of 40 in the power-spectrum, or even larger [11]. In Figure 2 we display the predicted gravitational lensing signal in B polarization calculated with CMBFAST for the cosmological model under consideration, reduced by a factor of 40 [10, 11]. Clearly, an improvement in the cleaning algorithm would make the second-order quadrupole polarization derived here the actual limitation to the detection of primordial gravitational waves through B-polarization of the CMB if  $r < \text{few} \times 10^{-5}$ .

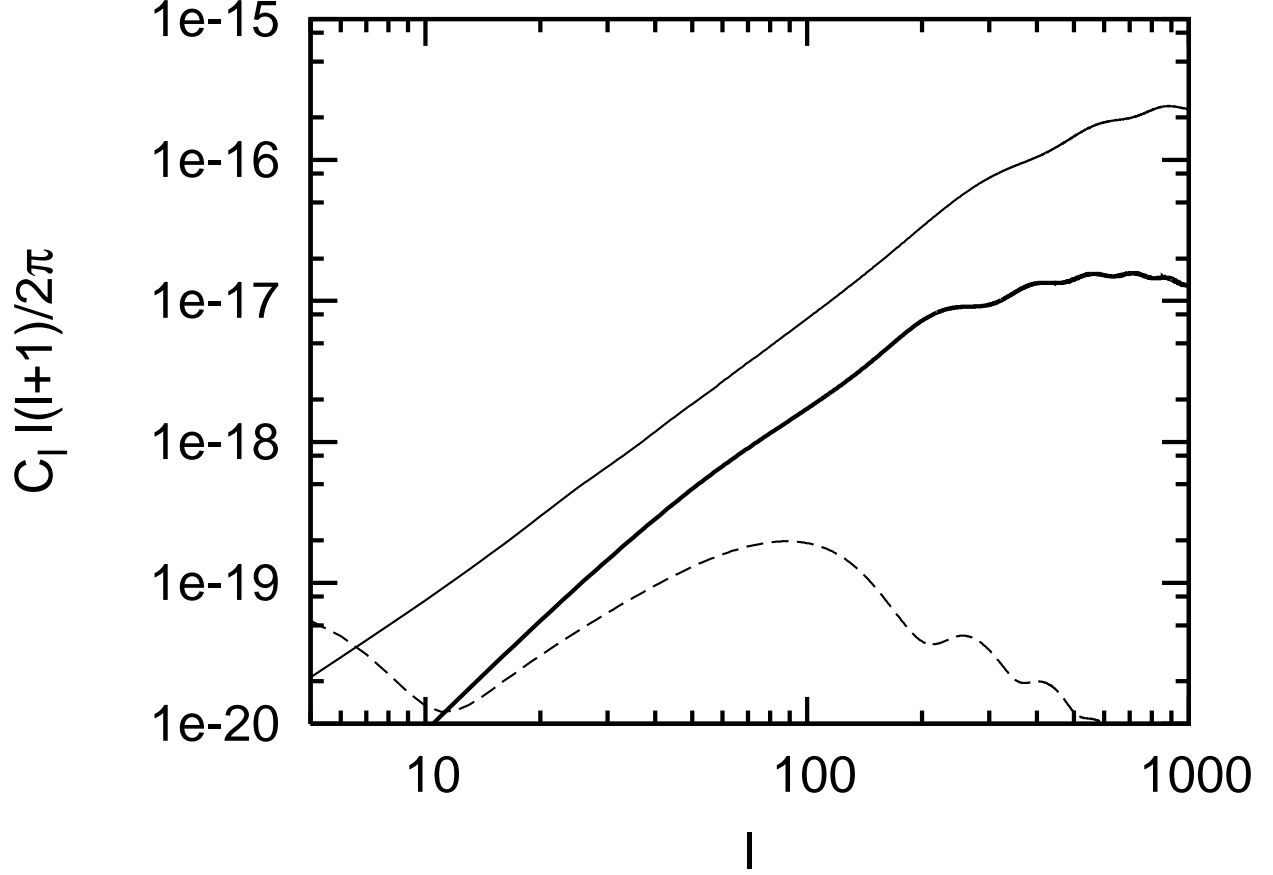


FIG. 2: Angular power spectrum of the total (vector plus tensor) B polarization generated at recombination by the second-order quadrupole (thick solid line) compared with that induced by primordial gravitational waves (dashed line) with tensor to scalar ratio  $r = 10^{-5}$  in a flat  $\Lambda$ CDM model. The thin solid line is the signal due to gravitational lensing cleaned by a factor 40.

### Acknowledgments

We wish to thank Wayne Hu for e-mail exchanges.

- 
- [1] U. Seljak and M. Zaldarriaga, Phys. Rev. Lett. **78**, 2054 (1997); *ibid*, Phys. Rev. D **55**, 1830 (1997).
  - [2] M. Kamionkowski, A. Kosowsky and A. Stebbins, Phys. Rev. Lett. **78**, 2058 (1997); *ibid*, Phys. Rev. D **55**, 7368 (1997).
  - [3] M. Amarie, C. Hirata and U. Seljak, Phys. Rev. D **72**, 123006 (2005); L. Page *et al.* [WMAP Collaboration], arXiv:astro-ph/0603450; F. Stivoli, C. Baccigalupi, D. Maino and R. Stompor, Mon. Not. R. astr. Soc. **372**, 615 (2006).
  - [4] W. Hu, M. M. Hedman and M. Zaldarriaga, Phys. Rev. D **67**, 043004 (2003)
  - [5] M. Zaldarriaga and U. Seljak, Phys. Rev. D **58**, 023003 (1998)
  - [6] L. Knox and Y. S. Song, Phys. Rev. Lett. **89**, 011303 (2002).
  - [7] M. Kesden, A. Cooray and M. Kamionkowski, Phys. Rev. Lett. **89**, 011304 (2002).
  - [8] M. Kesden, A. Cooray and M. Kamionkowski, Phys. Rev. D **67**, 123507 (2003)
  - [9] W. H. Kinney, arXiv:astro-ph/0307005.
  - [10] C. M. Hirata and U. Seljak, Phys. Rev. D **68**, 083002 (2003)
  - [11] U. Seljak and C. M. Hirata, Phys. Rev. D **69**, 043005 (2004)
  - [12] N. Bartolo, S. Matarrese and A. Riotto, JCAP **0701**, 019 (2007) [arXiv:astro-ph/0610110].
  - [13] T. Pyne and S. M. Carroll, Phys. Rev. D **53**, 2920 (1996); S. Mollerach and S. Matarrese, Phys. Rev. D **56**, 4494 (1997); N. Bartolo, S. Matarrese and A. Riotto, Phys. Rev. Lett. **93**, 231301 (2004); K. Tomita, Phys. Rev. D **71**, 083504 (2005); K. Tomita, Phys. Rev. D **72**, 043526 (2005); K. Tomita, Phys. Rev. D **72**, 103506 (2005) [Erratum-*ibid*. D **73**, 029901 (2006)]; N. Bartolo, S. Matarrese and A. Riotto, JCAP **0605**, 010 (2006).

- [14] N. Bartolo, S. Matarrese and A. Riotto, JCAP **0606**, 024 (2006) [arXiv:astro-ph/0604416].
- [15] S. Mollerach, D. Harari and S. Matarrese, Phys. Rev. D **69**, 063002 (2004).
- [16] W. Hu, Astrophys. J. **529**, 12 (2000).
- [17] S. Mollerach, Phys. Rev. D **42** (1990) 313; K. Enqvist and M. S. Sloth, Nucl. Phys. B **626**, 395 (2002); D. H. Lyth and D. Wands, Phys. Lett. B **524**, 5 (2002); T. Moroi and T. Takahashi, Phys. Lett. B **522**, 215 (2001) [Erratum-ibid. B **539**, 303 (2002)].
- [18] E. Bertschinger, in Proc. Les Houches School, Session LX, ed. R. Schaeffer (Netherlands:Elsevier) (1995).
- [19] S. Matarrese, S. Mollerach and M. Bruni, Phys. Rev. D **58**, 043504 (1998).
- [20] N. Bartolo, S. Matarrese and A. Riotto, JHEP **0404**, 006 (2004).
- [21] R. A. Sunyaev and Y. B. Zeldovich, Mon. Not. Roy. Astron. Soc. **190**, 413 (1980).
- [22] W. T. Hu, PhD thesis *Wandering in the background: A Cosmic microwave background explorer*, arXiv:astro-ph/9508126.
- [23] M. Zaldarriaga, D. D. Harari, Phys. Rev. D **52**, 3276 (1995).
- [24] W. Hu and M. White, Phys. Rev. D **56**, 596 (1997).
- [25] W. Hu, U. Seljak, M. White and M. Zaldarriaga, Phys. Rev. D **57** 3290 (1998).
- [26] S. Dodelson and J. M. Jubas, Astrophys. J. **439**, 503 (1995).
- [27] W. Hu and N. Sugiyama, Astrophys. J. **444**, 489 (1995).
- [28] H. V. Peiris *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **148**, 213 (2003)
- [29] D. N. Spergel *et al.* [WMAP Collaboration], arXiv:astro-ph/0603449.
- [30] U. Seljak and M. Zaldarriaga, Astrophys. J. **469**, 437 (1996).